Maxwell and the Electromagnetic Wave Equation*

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Maxwell's three derivations of the wave equations in electromagnetic theory are discussed and displayed on three charts.

ONE well-known fact of electromagnetic theory is that James Clerk Maxwell showed that a consequence of electromagnetic theory was that the fields obeyed wave equations and that these electromagnetic waves had as propagation velocity precisely the velocity of light. Although this result is known, Maxwell's starting points and his method of derivation are not well known; Maxwell's basic equations differed from those in use today, so it is not surprising that the derivations are also different.

The purpose of this article is to exhibit and discuss Maxwell's derivations of the wave equations from the basic equations of electromagnetic theory in two papers and in the *Treatise on Electricity and Magnetism*.[1,2] Most of the burden of the exposition is in the three charts, Figs. 1–3, one corresponding to each derivation. Therefore, our discussion is confined to a description of the charts, to pointing out additional features not seen on the charts, and to comparing the different methods. Because of the charts we can make our verbal description brief.

The three charts are logical flow diagrams; the arrows coming into a box indicate the statements Maxwell explicitly uses in the proof of the statement in that box. Other theorems which a reader today might consider necessary are not indicated, and occasionally the decision as to when to display a line is ambiguous. It should be emphasized that the notation used in the charts is not that of Maxwell. The notation in the papers is a component notation, and in the *Treatise* both component and quaternion notations are used; the notation of the charts is the modern vector notation for electromagnetic theory with some additions. In the 1865 paper and in the *Treatise*, Maxwell uses the "total current," the sum of conduction and displacement currents, and this current has an explicit symbol. Since the total current occurs in several fundamental equations, it is necessary for us to have a symbol for it, and we have used $\mathbf{j}_t$. There is some danger in this "translation" from an older notation into a newer one, and the careful reader is referred to the original material for complete details. The relations between Maxwell's symbols (other than quaternions) and those in the charts are given in Table I.

Before considering the three logical structures, we note that Maxwell mentions the electromagnetic theory of light first in his 1861–1862 paper "On Physical Lines of Force."[3] But the results there cannot be said to be *derived* from fundamental equations; rather, electromagnetic waves are a consequence of the mechanical model Maxwell uses in that paper. Hence we have no chart concerning this paper.

I. "A DYNAMICAL THEORY OF THE ELECTROMAGNETIC FIELD"

Maxwell's 1865 paper in the *Philosophical Transactions of the Royal Society* is the first full-blown treatment of electromagnetic theory.[4] This paper is carefully organized; the electromagnetic discussion is based on 20 equations

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[1] James Clerk Maxwell, "A Dynamical Theory of the Electromagnetic Field," in *Scientific Papers of James Clerk Maxwell*, W. D. Niven, Ed. (Librarie Scientifique J. Hermann, Paris, 1927), Vol. 1, pp. 526–597; "On a Method of Making a Direct Comparison of Electrostatic with Electromagnetic Force; with a Note on the Electromagnetic Theory of Light," *ibid.*, Vol. 2, pp. 125-143.

[2] James Clerk Maxwell, *Treatise on Electricity and Magnetism* (The Clarendon Press, Oxford, England 1881), 2nd ed.

[3] Ref. 1, Vol. 1, pp. 451-514.

[4] Ref. 1, Vol. 1, pp. 526–597.

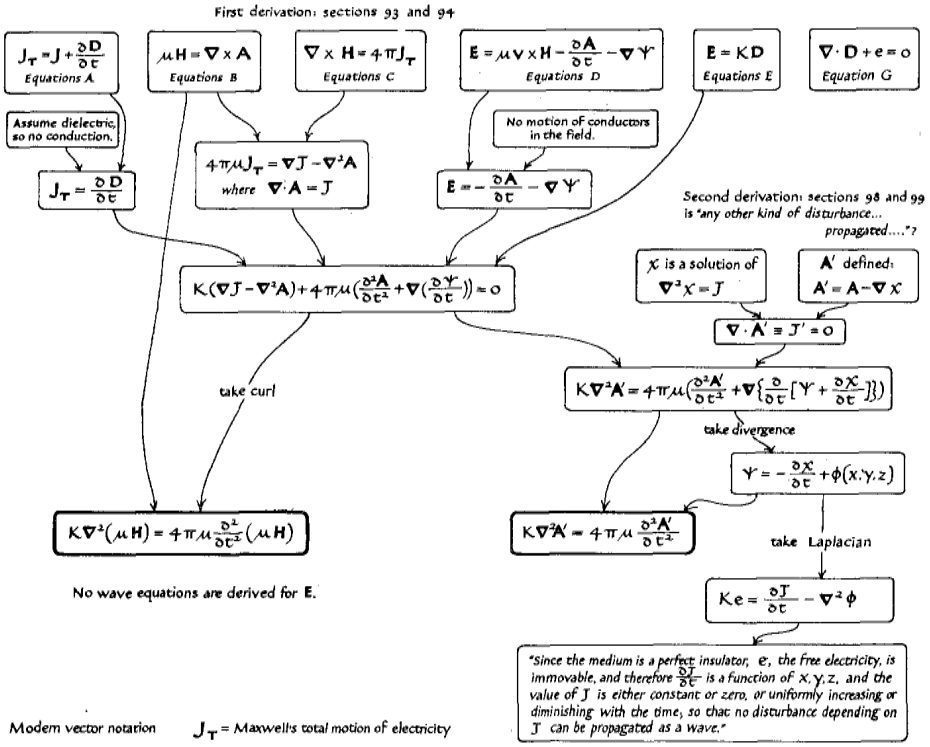
First derivation: sections 93 and 94
Equations A
Equations B
Equations C
Equations D
Equations E
Equation G
Assume dielectric, so no conduction.
No motion of conductors in the field.
where
Second derivation: sections 98 and 99 is "any other kind of disturbance... propagated...."?
is a solution of
defined:
take curl
take divergence
take Laplacian
No wave equations are derived for E.
"Since the medium is a perfect insulator, e, the free electricity, is immovable, and therefore is a function of x, y, z, and the value of J is either constant or zero, or uniformly increasing or diminishing with the time, so that no disturbance depending on J can be propagated as a wave."
Modern vector notation
= Maxwell's total motion of electricity

FIG. 1. Maxwell's derivation of the wave equations in "A Dynamical Theory of the Electromagnetic Field."

involving 20 variables, the names of the equations and the names of the variables both carefully listed.

Part 4 is called the "Electromagnetic Theory of Light." As can be seen from Fig. 1, Maxwell derives the wave equations twice from the basic

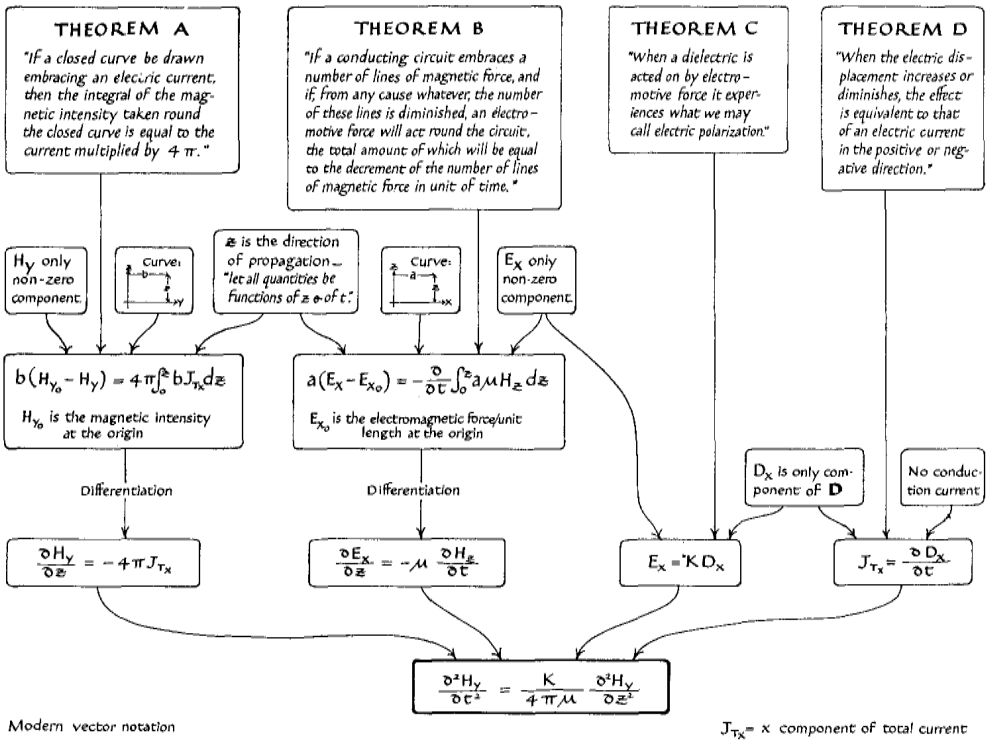
THEOREM A
"If a closed curve be drawn embracing an electric current, then the integral of the magnetic intensity taken round the closed curve is equal to the current multiplied by 4π."
THEOREM B
"If a conducting circuit embraces a number of lines of magnetic force, and if, from any cause whatever, the number of these lines is diminished, an electromotive force will act round the circuit, the total amount of which will be equal to the decrement of the number of lines of magnetic force in unit of time."
THEOREM C
"When a dielectric is acted on by electromotive force it experiences what we may call electric polarization."
THEOREM D
"When the electric displacement increases or diminishes, the effect is equivalent to that of an electric current in the positive or negative direction."
only non-zero component
curve:
is the direction of propagation — "let all quantities be functions of z & of t."
Differentiation
is the magnetic intensity at the origin
is the electromagnetic force/unit length at the origin
is only component of
No conduction current
Modern vector notation
= x component of total current

FIG. 2. Maxwell's derivation of the wave equation in the "Note on the Electromagnetic Theory of Light."

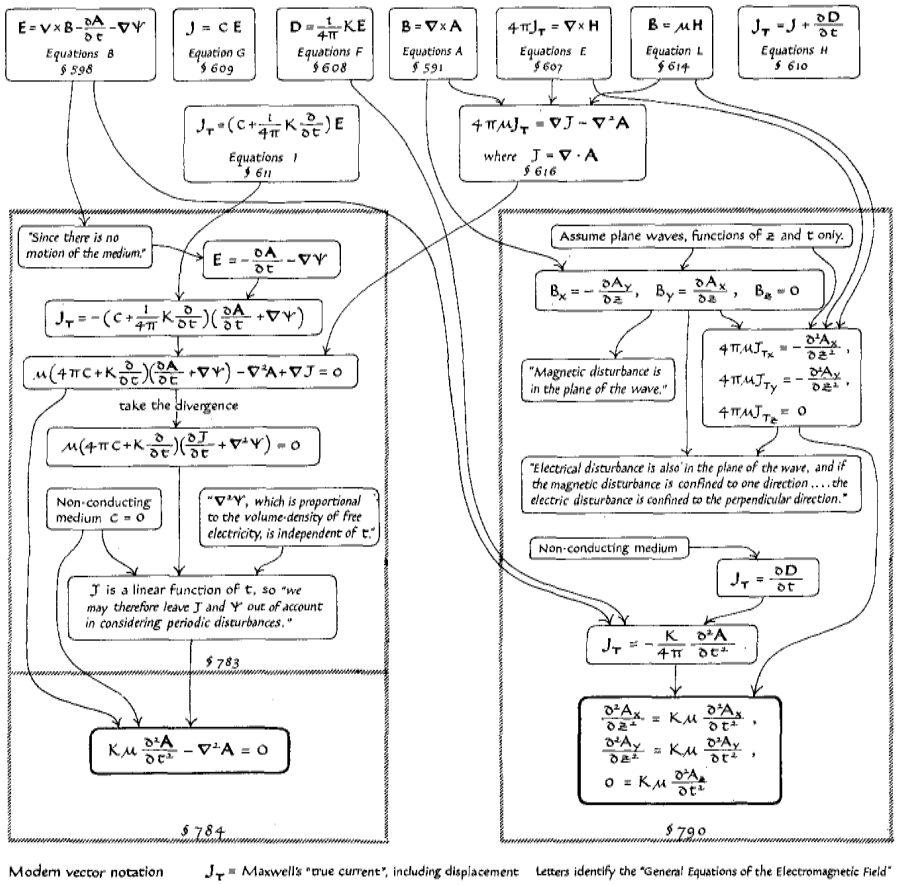
Equations B § 598
Equation G § 609
Equations F § 608
Equations A § 591
Equations E § 607
Equation L § 614
Equations H § 610
Equations I § 611
where § 616
"Since there is no motion of the medium."
Assume plane waves, functions of z and t only.
take the divergence
"Magnetic disturbance is in the plane of the wave."
"Electrical disturbance is also in the plane of the wave, and if the magnetic disturbance is confined to one direction .... the electric disturbance is confined to the perpendicular direction."
Non-conducting medium
Non-conducting medium C = 0
"∇²Ψ, which is proportional to the volume-density of free electricity, is independent of t."
J is a linear function of t, so "we may therefore leave J and Ψ out of account in considering periodic disturbances."
§ 783
§ 784
§ 790
Modern vector notation
J_T = Maxwell's "true current", including displacement
Letters identify the "General Equations of the Electromagnetic Field"

FIG. 3. Maxwell's derivation of the wave equations in the Treatise on Electricity and Magnetism, 2nd Ed.

equations of electromagnetic theory. The first of these derivations leads to the wave equations for the magnetic field, although the details heavily involve the vector potential.

Immediately after obtaining these wave equations and noting the velocity, Maxwell says:

> This wave consists entirely of magnetic disturbances, the direction of magnetization being in the plane of the wave. No magnetic disturbance whose direction of magnetization is not in the plane of the wave can be propagated as a plane wave at all.

It is difficult to establish how much weight to put on Maxwell's use of "entirely." If one interprets it literally, Maxwell is saying that the electromagnetic wave is a traveling *magnetic* disturbance, but not an electrical disturbance. Indeed, in this paper, as opposed to the 1868 paper and to the *Treatise*, there is no mention of propagation of *electric* field quantities; Maxwell does not derive a corresponding wave equation involving the electric field. Maxwell's 20 fundamental equations in this paper do not have the symmetry between electric and magnetic quantities which is characteristic of present treatments, a symmetry imposed by Heaviside and Hertz; so it is not obvious in Maxwell's form that the components of electromotive force should satisfy a wave equation. Hence it is conceivable that Maxwell did not know in 1865 that the electric field was also propagated.

After comparison of velocity, Maxwell returns to a second consideration of electromagnetic quantities obeying a wave equation. He says,

> Let us now go back upon the equations in which the quantities J and V occur, to see whether any other kind of disturbance can be propagated through the medium depending on these quantities which disappeared from the final equations.

The modern student of electromagnetic theory knows that to obtain a wave equation for the vector potential, Maxwell's intent here, demands what we would today consider an auxiliary "gauge" condition on a potential; so it is of interest to see how Maxwell treats this situation.

TABLE I. Translation table for the charts.

"Dynamical Theory"	*Treatise on Electricity and Magnetism*	Chart
P, Q, R	P, Q, R	E
$\frac{dx}{dt}\ \frac{dy}{dt}\ \frac{dz}{dt}$	$\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$	v
. . .	a, b, c	B
F, G, H	F, G, H	A
ψ	ψ	ψ
p', q', r'	u, v, w	j_t
p, q, r	p, q, r	j
f, g, h	f, g, h	D
α, β, γ	α, β, γ	H
μ	μ	μ
. . .	∇^2	$-\nabla^2$

"Note on the Electromagnetic Theory of Light"		Chart
Magnetic intensity in y direction	B	H_y
Total current	p	j_t
Electromotive force in x direction	P	E_x
Coefficient of Magnetic induction	μ	μ
Electric elasticity	K	K
Electric displacement (x component)	f	D_x

One should note that Maxwell is not using the *name* vector potential here, a name used for the first time in print in the *Treatise on Electricity and Magnetism*, but is calling this quantity the electromagnetic momentum. Earlier, he had used the term *electrotonic state*, after Faraday.

It can be seen from the chart that Maxwell's basic tactic, in a language familiar today but that he is not using, is to make a gauge transformation on the vector potential such that the new potential has zero divergence. Maxwell's J, not to be confused with our modern current symbol, is the divergence of the vector potential. Then the new components all satisfy the wave equation. Maxwell tries to justify this gauge transformation, a "justification" presenting some difficulty to him. The printer's proof of the paper says,[5]

> Here J is either zero or it continually increases or diminishes with the time, if e remains constant, which no physical quantity can do. Hence J is zero, and the only disturbance propagated is that indicated by F', G', H' [the transformed vector potential], which is wholly transversal.

In the published paper Maxwell no longer argues that J is zero on physical grounds, but that "no disturbance depending on J can be propagated as a wave."

II. "NOTE ON THE ELECTROMAGNETIC THEORY OF LIGHT"

Maxwell's 1868 "Note on the Electromagnetic Theory of Light" is both the most modern of Maxwell's derivations of the wave equations and at the same time the one that had the least direct effect on the history of physics.[6] As I have commented elsewhere, this paper was scarcely known to Maxwell's immediate followers. J. J. Thomson apparently rediscovered it after Heaviside and Hertz had rewritten Maxwell's equations in the circuital form. However, there is no evidence, as far as I am aware, that it was known before Thomson's rediscovery to either Heaviside or Hertz. In this paper Maxwell was trying to delineate his version of the electromagnetic theory of light from related theories of Riemann, Weber, and Lorenz. Hence, he strove for the simplest statement of the fundamental equations and for the simplest derivation of the wave equations.

The derivation is strictly a nonvector derivation; after the verbal statements of the four theorems, Maxwell assumes that all the quantities depend only on z and t. The only quantity for which a wave equation is derived is H_y, but based on the sinusoidal solution for H_y he exhibits waves for j_{tx} (the total current), E_x, and D_x. So by 1868 Maxwell definitely knew that the electric field is also propagated.

Heaviside gave, at the beginning of the third volume of his *Electromagnetic Theory*,[7] a derivation closely related to this, also based on the circuital form of the equations. The main difference is that Heaviside assumes that he is deal-

[5] P. F Cranefield, "Clerk Maxwell's Corrections to the Page Proofs of 'A Dynamical Theory of the Electromagnetic Field'," Ann. Sci. 10, 360 (1954).

[6] Ref. 1, Vol. 2, pp. 125–143.

[7] O. Heaviside, *Electromagnetic Theory* (Benn Brothers, Ltd., London, 1922), Vol. 3, Chap. IX.

ing at the beginning with a wavefront advancing with velocity v, such that the field is zero on one side of the wave and a constant nonzero on the other side. It is interesting to note that this derivation has been rediscovered time and time again, and is used in several textbooks without realization that it was originated by Heaviside.

This is the only one of Maxwell's derivations *not* involving the vector potential, as, unlike Maxwell's other bases, the circuital form of the basic equations given in the four theorems in this paper does not involve the vector potential. As a consequence, the arguments about what we now call the gauge condition, present in the other derivations, are missing here.

III. TREATISE ON ELECTRICITY AND MAGNETISM

A glance at the diagram shows that the structure of Maxwell's derivations of the wave equations in the *Treatise on Electricity and Magnetism*[2] is related, both in starting point and in logical development, to that in "A Dynamical Theory of the Electromagnetic Field," and has little resemblance to the proof in the "Note on the Electromagnetic Theory of Light." As one might expect in a *Treatise,* the material is much more scattered than in the paper. The general equations of the electromagnetic field are developed around Article 600, while the derivations of the wave equations occur about Article 780. However, Maxwell has prepared for the deduction in earlier articles.

In spite of the similarities between the 1865 paper, "A Dynamical Theory of the Electromagnetic Field," and the *Treatise,* there are interesting differences. First, for reasons not clear to me, Maxwell does not use the same letter names for the basic equations in the two sources. Thus the relation between the magnetic field and the curl of the vector potential is equation B in the 1865 paper, but is equation A in the *Treatise;* the relation between the potentials and the electric field is equation D in the "Dynamical Theory," while it is equation B in the *Treatise.* I do not see any motivation for Maxwell's change in the names. A second difference obvious from the chart is that Maxwell is using a magnetic vector, equivalent to our $\mathbf{B}$, in place of our $\mu\mathbf{H}$ in the 1865 paper. This is a minor change, but does make the notation for magnetic fields more similar to that for electric fields; in the *Treatise* two vectors are used to describe each.

The derivations are in component notation rather than in the newly introduced quaternion notation. As in the "Dynamical Theory," the vector potential is a major concept. The exact details differ, but again it is necessary to introduce an auxiliary condition on the potential; Maxwell argues that the divergence of the vector potential can be set equal to zero, a constant, or a linear function of time. (Here Maxwell is using the name *vector potential.*) Because Maxwell views the vector potential as a fundamental quantity in electromagnetic theory, he considers the choice of a gauge differently than we do today. He feels that it needs to be justified as an assumption in addition to the basic equations, and not simply assumed. The fact that his justifications differ, and that none are too convincing, indicates that he may have had difficulties with this problem.

It is difficult to convince oneself that Maxwell has in any sense improved his argument for the condition on J in the *Treatise,* as compared with the argument in the 1865 paper. The argument in the *Treatise* relies on an equation,

$$\nabla^2 \psi = -\rho,$$

which is not one of the fundamental equations or derivable from them, and which is not generally true. Equation J relates the charge density to the change in the electric field, and when the result is expressed in terms of the potentials there is a term involving the vector potential. The chart for the "Dynamical Theory" shows that Maxwell *has* previously used the more general relation applicable here, the relation in the second box from the bottom on the righthand side. What is missing in the *Treatise* is the j term, necessary unless J has *already* been taken to be zero or constant. Further, the possibility that J is a linear function of time is not strong enough. That Maxwell has varied the argument perhaps shows that he is queasy about the question of the gauge condition. The peculiarities of these arguments may have influenced Heaviside and Hertz in deciding that electro-

magnetic theory was better without the potentials, although this conjecture receives little direct support from their writings.

Possibly, as was not the case for the 1865 paper, he is fully aware that both the electric and magnetic fields obey wave equations and he explicitly mentions the propagation of both. He draws the famous diagram showing the electric displacement and magnetic field at right angles to each other for the sinusoidal case. But analytically, he develops the general plane-wave solution, not just the sinusoidal solution. Again, Maxwell follows this derivation of the simple wave equation with a study of the propagation of waves in crystals.

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